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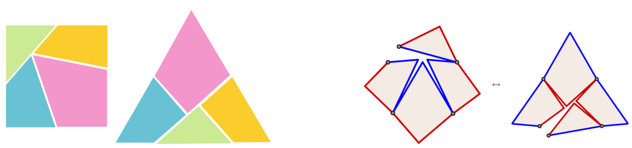
1 TWIST-HINGE DISSECTIONS

A *geometric dissection* is a partition of a shape into pieces that can be rearranged to form another shape (Figure 1a). The Wallace-Bolyai-Gerwien theorem says that any two polygons of equal area has such a partition [Gardner 1985], but the number and shape of the pieces can be physically intractable to manufacture, and finding the minimum number of pieces, either exactly or to a constant factor approximation, is NP-hard [Bosboom et al. 2015]. The graphics community has shown interest in this problem through a method to compute dissections for lattice-based polygons [Zhou and Wang 2012], and approximate dissections, where the pieces does not have to reproduce the shapes exactly [Duncan et al. 2017].



(a) A geometric dissection.

(b) A hinged dissection.

Fig. 1. Two examples of dissections. In a (geometric) dissection one can rearrange the pieces freely to turn one shape into the other. In a hinged dissection the pieces are connected with hinges at the corners, and they are allowed to swing to go in between the shapes. (b) is from [Akiyama et al. 2020].

A variant of dissections is *hinge dissections*, where the pieces are connected by a hinge at the corners (Figure 1b). By turning the pieces around their hinges one can rearrange one shape into another. Abbott et al. [2012] showed that any finite collection of polygons of equal area has a hinged dissection. By making an additional requirement we get a *reversible* hinged dissection: here the pieces are connected in a simple path (as opposed to a tree), and the rotation at every hinge goes from being clockwise in one configuration to counter-clockwise for the other. Li et al. [2018] proposed a method for computing approximate reversible hinged dissections.

In this project we look at another variation called *twist-hinge dissections*. This is a dissection where adjacent pieces are connected by a rod at their common edge, allowing rotation around the rods axis (Figures 2 and 3). The rotation causes the pieces to temporarily go out of plane, but by rotating $k \cdot 180$ degrees for some integer k we end up with another planar configuration.

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1.1 Applications

Dissections in general have many practical applications, for instance (1) multi-purpose and re-configurable furniture [Song et al. 2017]; (2) objects can be stored and shipped in a compact tillable shape, and reconfigured to its final shape; (3) programmable materials where logic is encoded in the shape; reconfigured the dissection and changing the shape reprograms the material; (4) motorized and programmable hinges can be used for robotics; (5) recreational puzzles.

1.2 Open Problems

Since the literature is sparse on this topic there are many natural questions that are still open.

1.2.1 Twist-Hinge Dissections. To our knowledge there are no universality results on twist-hinge dissections. Which shapes have such a dissection? Which natural shape pairs admit a nice approximation for a twist-hinge dissection? How do we compute the dissection, and how do we ensure that the pieces are possible to fabricate?

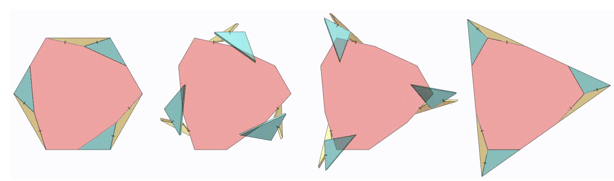

Fig. 2. An example of a twist-hinge dissection in which a hexagon is transformed into an equilateral triangle. From [Frederickson 2010].

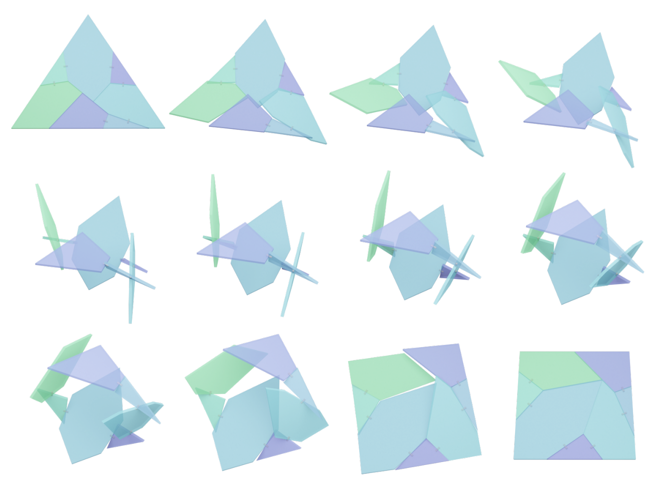

Fig. 3. The evolution of a twist-hinge dissection of an equilateral triangle morphing into a square. Note how the purple pentagon at the base of the triangle is almost purely translated from its initial to final position; this is due to it being an even number of hinges away from the fixed hexagon at the tip of the triangle, as well as the rotations of its hinges canceling out. All pieces with an odd number of hinges to the fixed hexagon are flipped.

1.2.2 Rotation Scheduling. If we rotate one hinge at a time it is often the case that the intermediate flat configurations in between the rotations is overlapping with itself, and if we try to rotate multiple hinges at the same time the pieces can collide mid-air. The green and purple pieces in Figure 3 are nearly colliding in the middle row frames, and indeed, it is possible to adjust the rotations slightly so that they do collide. Figure 4 shows a sequence of rotations where all intermediary states are non-overlapping, but it is easy to find examples of hinges that would make pieces overlap had they been turned. Scheduling the rotations to guarantee a collision-free trajectory is an interesting problem. One could also minimize the time spent performing the rotations.

1.2.3 Physically Aware Dissections. In the scheduling problem above we are already given the dissection. Complex dissections might require adding additional cuts with hinges that are only partially rotated, and then rotated back to its original position, to avoid collisions. Adding in physical based constraints to the problem formulation of deciding the dissection could be valuable.

If the hinges are motorized, the torque they can apply is bounded, so it would be interesting to compute a rotation sequence that would minimize the maximal torque required for the transformation. A slight variation of this problem is to look at the maximal stresses on the hinges, or the pieces themselves, to ensure that they do not break during the rotation. This would be natural to combine with the collision-free requirement so that the trajectories are feasible in the real world.

1.3 Attacks

Here are some attack angles on the sub-problems from Section 1.2 that would be good places to start.

1.3.1 A Smooth Map. Given a dissection we can reason about its behavior when the pieces are rotated around the hinges. The connectivity of the pieces form a tree, and by fixing an arbitrary piece as the root we can compute affine transformations consisting of reflections around points and axes for the remaining pieces. The final piece positions are smooth functions of the positions of the dissection cuts, since they are compositions of affine transformations. Maybe it is possible to formulate an optimization problem to approximate a second shape based on this.

1.3.2 A Bottom Up Approach. There are universality theorems for regular dissections and hinged dissections in the literature, and a necessary and sufficient condition for two shapes to have reversible hinged dissections has also been found [Akiyama et al. 2020]. To my knowledge there are no such results for twist-hinges. It would be natural to try to characterize the space of shapes that admit a twist-hinged dissection, even though finding a complete characterization is not the main objective of the project. This could give us intuition of the space, as well as heuristics to guide a search or algorithms that work for shapes under certain constraints.

1.3.3 Greedy Decomposition. The theoretical results in dissection often involves finding *gadgets* that implements a certain behavior, like moving a hinge from one side of a polygon to another. If we can find small gadgets for twist-hinges we can triangulate the target shapes and introduce gadgets to move the triangles where we need them to move. This will very likely make the number of pieces far too large and the pieces too small, but by approximating the shapes we can greedily merge pieces into larger but less accurate pieces until we have a sufficient number of pieces.

1.3.4 Collision Testing. Given all joint angles $\{\theta_i\}$ we can compute the planes for all pieces. If two pieces are intersecting that means that their planes intersect such that the intersection line is intersecting the piece polygons in their planes, in the same place. For the thickened pieces we can compute a threshold $\epsilon(w)$ as a function of the width w, and adjust the line-polygon intersection test so that a non-intersection implies that the pieces does not intersect.

To compute which hinges should rotate at the same time we can impose the constraint that they should rotate synchronously with the same angular speed. Now we need to compute whether any pair of pieces intersect when a subset of the hinges are rotated by any of the angles $\theta \in [0, \pi]$. One approach is to bound the angle increment required when going from a non-intersecting configuration to a intersecting one, by bounding the speed at which the plane-plane intersection lines move, and use this to search for an intersecting configuration. If the bound is sufficiently tight this should be feasible, since we can check all piece pairs in parallel.

1.3.5 Rotation Scheduling. To find which rotations we can perform at the same time we could use a greedy approach where we impose an arbitrary ordering $\{\theta_1, \ldots, \theta_n\}$ of all the rotations. Then we go through the rotations in order and perform the ones that are feasible so that the rotated configuration doesn't self intersect during or after the rotations. The ordering of the rotations will decide the quality of the scheduling, in the sense that a bad ordering will give us a bad schedule, but the best ordering will give us the optimal schedule. There might exist good heuristics for finding orderings with provable guarantees on the resulting schedule.

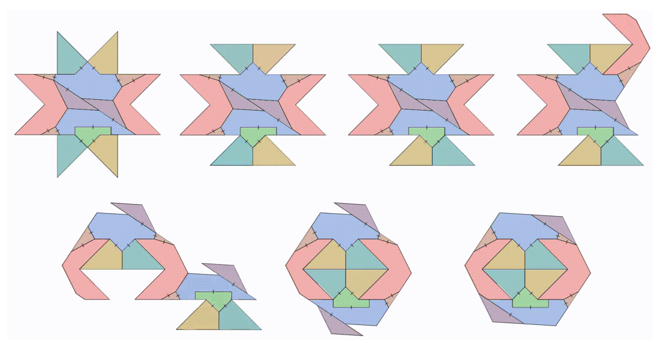

Fig. 4. A complex example of a twist-hinge dissection. Here a $\{8/3\}$-star is transformed into a regular hexagon. From [Frederickson 2007].

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